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Magic $N = 2$ supergravities from hyper-free superstrings

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Abstract: We show by explicit construction the existence of various four dimensional models of type II superstrings with $N = 2$ supersymmetry, purely vector multiplet spectrum and no hypermultiplets. Among these, two are of special interest, at the field theory level they correspond to the two exceptional $N = 2$ supergravities of the magic square that have the same massless scalar field content as pure $N = 6$ supergravity and $N = 3$ supergravity coupled to three extra vector multiplets. The $N = 2$ model of the magic square that is associated to $N = 6$ supergravity is very peculiar since not only the scalar degrees of freedom but all the bosonic massless degrees of freedom are the same in both theories. All presented hyper-free $N = 2$ models are based on asymmetric orbifold constructions with $\mathcal{N} = (4, 1)$ world-sheet superconformal symmetry and utilize the 2d fermionic construction techniques. The two exceptional $N = 2$ models of the magic square are constructed via a "twisting mechanism" that eliminates the extra gravitini of the $N = 6$ and $N = 3$ extended supergravities and creates at the same time the extra spin-$\frac{1}{2}$ fermions and spin-1 gauge bosons which are necessary to balance the numbers of bosons and fermions. Theories of the magic square with the same amount of supersymmetry in three and five space-time dimensions are constructed as well, via stringy reduction and oxidation from the corresponding four-dimensional models.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Extended Supersymmetry, Supergravity Models.

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Contents

1. Introduction

In four dimensions there is an exceptional family of $N = 2$ supergravities [1] which are known to be in correspondence with the symmetric spaces of the "magic square" of Freudenthal-Rozenfeld-Tits [2]. Two of the four $N = 2$ exceptional theories in four dimensions are associated to the Jordan algebras $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{H}}$. They have the additional remarkable property that they share the same scalar field contents as some supergravity models with more supersymmetry. Their scalar manifolds

$$\mathcal{M}_3 = \frac{\mathrm{SU}(3,3)}{S(\mathrm{U}(3) \times \mathrm{U}(3))}, \qquad \mathcal{M}_6 = \frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)} \tag{1.1}$$

appear also in $N = 3$ supergravity with 3 vector supermultiplets and in (pure) $N = 6$ supergravity, respectively.

The $N = 2$ model which is associated to $N = 6$ supergravity appears to be most peculiar. Not only the scalar fields of both theories are the same but all the other bosonic degrees of freedom are also the same. This remarkable property motivated us to realize this exceptional $N = 2$ theory at the superstring level. In order to proceed in this direction we found useful to focus our attention more generally on the construction of $N = 2$ theories with purely vector multiplet spectrum i.e. without hypermultiplets. As a consequence, the scalar fields will all belong to vector multiplets and parameterize a projective special Kähler manifold [3 – 5]. All theories of the "magic square" belong to the class of hyper-free theories.

In this work we construct four dimensional type II superstring backgrounds with $N = 2$ space-time supersymmetry.[1] The $N = 2$ hyper-free models are constructed via asymmetric orbifolds utilizing the free 2d-fermion construction [6, 7]. Due to the left-right asymmetric construction the universal axion-dilaton pair, S, will not be part of a hypermultiplet, as would be the case for type II geometric compactifications on a Calabi-Yau manifold. Instead, S will belong to a vector multiplet as in the heterotic-like compactifications. This is linked to the fact that supersymmetry is realized in an asymmetric way from the world-sheet point of view [6, 8, 9], as in the heterotic case or in the asymmetric constructions of type II obtained in [10, 11]. Instead of having $\mathcal{N} = (2, 2)$ superconformal models as is the case for a Calabi-Yau compactification we rather have $\mathcal{N} = (4, 1)$ superconformal symmetry on the world-sheet [12]. We may note that the c-map correspondence exchanges hypermultiplets and vector multiplets, it is a T duality in 3d conjugated by scalar/vector dualities, we shall return to this in due course.

In section 2 we start with the construction of the "$S-$minimal" $N = 2$ type II superstring model which contains a single minimally coupled vector multiplet, S, associated to the axion-dilaton pair. The $S-$minimal theory is universal in the sense that it will be part of the spectrum of all the other more complex hyper-free models.

In section 3 we recall the properties of some exceptional $N = 2$ supergravities [1] which are known to be in correspondence with the projective special Kähler symmetric spaces of the "magic square" of Freudenthal-Rozenfeld-Tits [2].

In section 4 and 5 we construct as type II superstrings the two $N = 2$ theories of the "magic square" associated to the $N = 6$ and $N = 3$ supergravity theories respectively. We first recall how to realize the associated theories with higher supersymmetry ($N = 6$ and $N = 3$) and then we introduce a new mechanism which reduces supersymmetry *without* changing the scalar content of the models. Its sole effect will be to somehow replace the gravitini that we want to get rid off, by spin-1/2 fermions and spin-1 gauge bosons, thereby giving us precisely the spectrum of the $N = 2$ theories we are looking for. Two extra theories of the magic square in three space-time dimensions, $\mathcal{M}_6^{D=3} = \frac{E_{7(-5)}}{\mathrm{SU}(2)\times\mathrm{SO}(12)}$, and $\mathcal{M}_3^{D=3} = \frac{E_{6(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(6)}$ are constructed in section 4 and 5 by stringy dimensional reduction from the corresponding four dimensional magic theories. Furthermore, by oxidation in five space-time dimensions the construction of $\mathcal{M}_6^{D=5} = \frac{\mathrm{SU}^*(6)}{\mathrm{USP}(6)}$ of the magic square is also obtained. In our stringy set-up, there is an obstruction to define $\mathcal{M}_3^{D=5} = \frac{\mathrm{SL}(3,C)}{\mathrm{SU}(3)}$ since all six right-moving coordinates are twisted which prevent the oxidation procedure. Section 6 summarizes our results.

2. The minimal hyper-free theory

Our first theory is a type IIA four dimensional $N = 2$ model which contains the graviton supermultiplet and one additional vector multiplet in its massless spectrum. The

[1] Here and in the following $N = N_4$ corresponds to the number of space-time supersymmetries with respect to the four dimensional super-Poincaré algebra. It will sometimes be written as $N = N_L + N_R$ to recall the world-sheet chirality responsible for the supercharges

vector multiplet consists of the universal dilaton and axion fields. The model can be easily obtained via the fermionic construction [6–8]; the basis of sets $\{F, S, \overline{S}, b_1, \overline{b}_1, \overline{b}_2, \overline{b}_3\}$ is the main data relevant for the construction of the corresponding model which we call $\langle F, S, \overline{S}, b_1, \overline{b}_1, \overline{b}_2, \overline{b}_3\rangle$ but it implies also a choice of some generalized GSO projection coefficients (GGSO or discrete torsion). Here F is the set containing all the fermions of the model in the light-cone gauge, namely two world-sheet fermions ψ^μ, the 12 fermionized internal coordinates $\{y^I, w^I\}$ with $\partial X^I = y^I w^I, I = 1 \ldots 6$, their 6 world-sheet supersymmetric partners χ^I as well as all the corresponding right- moving 2d fermions. The sets S, $\overline{S}$ that define the left- and right- supersymmetric GSO projections are given by [6]:

$$S = \left\{\psi^\mu, \chi^{1,\ldots,6}\right\}, \qquad \overline{S} = \left\{\overline{\psi}^\mu, \overline{\chi}^{1,\ldots,6}\right\} . \tag{2.1}$$

The model defined by the sets $\{F, S, \overline{S}\}$ gives rise to the usual $N = 4 + 4$ supersymmetric background. In order to reduce the left- plus right-moving space-time supersymmetry from $N = 4 + 4$ to $N = 2 + 1$ resp. to $N = 2 + 0$, it is necessary to include some extra "supersymmetry breaking sets", $b_1, \overline{b}_1, \overline{b}_2$ resp. $b_1, \overline{b}_1, \overline{b}_2, \overline{b}_3$ that define left-right-asymmetric projections of the type:

$$(Z_2)_{\text{left}} \times (Z_2 \times Z_2)_{\text{right}}$$

resp.

$$(Z_2)_{\text{left}} \times (Z_2 \times Z_2 \times Z_2)_{\text{right}}$$

with

$$\begin{aligned}
b_1 &= \left\{ \psi^\mu,\ \chi^{1,2},\ y^{3,4},\ y^{5,6},\ y^1, w^1 \,|\, \overline{y}^5, \overline{w}^5 \right\} \\
\overline{b}_1 &= \left\{ \overline{\psi}^\mu,\ \overline{\chi}^{1,2},\ \overline{y}^{3,4},\ \overline{y}^{5,6},\ \overline{y}^1,\ \overline{w}^1 \,|\, y^5, w^5 \right\} \\
\overline{b}_2 &= \left\{ \overline{\psi}^\mu,\ \overline{\chi}^{3,4},\ \overline{y}^{1,2},\ \overline{w}^{5,6},\ \overline{y}^3,\ \overline{w}^3 \,|\, y^6, w^6 \right\} \\
\overline{b}_3 &= \left\{ \overline{\psi}^\mu,\ \overline{\chi}^{5,6},\ \overline{w}^{1,2},\ \overline{w}^{3,4},\ \overline{y}^6,\ \overline{w}^6 \,|\, y^2, w^2 \right\}
\end{aligned} \tag{2.2}$$

The above choices of the "supersymmetry breaking sets" define consistent models since they satisfy all the overlapping conditions necessary in the fermionic construction [6].

$$\begin{aligned}
&\overline{b}_1 \cap\ \overline{b}_2 = \left\{ \overline{\psi}^\mu,\ \overline{y}^{1,3} \right\}; \qquad && \overline{b}_1 \cap b_1 = \left\{ \overline{y}^5 \,|\, y^5 \right\}; \qquad && \overline{b}_2 \cap b_1 = \left\{ \overline{w}^5 \,|\, y^6 \right\} \\
&\overline{b}_3 \cap\ \overline{b}_1 = \left\{ \overline{\psi}^\mu,\ \overline{y}^6,\ \overline{w}^1 \right\}; \qquad && \overline{b}_3 \cap\ \overline{b}_2 = \left\{ \overline{\psi}^\mu,\ \overline{w}^{3,6} \right\}; \qquad && \overline{b}_3 \cap b_1 = \emptyset
\end{aligned} \tag{2.3}$$

and

$$\overline{b}_1 \cap\ \overline{b}_2 \cap\ \overline{b}_3 \cap b_1 = \emptyset. \tag{2.4}$$

The presence of some left- and right-fermionized coordinates, $\{y_i, w_i | \overline{y}_j, \overline{w}_j\}$, in the $\overline{b}_I, I = 1, 2, 3$ and b_1 guarantees the free-action of the Z_2's defining the asymmetric orbifolds [13] and thus the absence of massless states coming from the "twisted sectors". The models are defined completely once we specify the signs of the generalized GSO coefficients (GGSO). We have chosen to take minus one for all but one:

$$(-1)^{\ \overline{b}_1} = (-1)^{\ \overline{b}_2} = (-1)^{\ \overline{b}_3} = (-1)^{b_1} = -1\,, \tag{2.5}$$

in addition to the choices

$$(-1)^F = 1, \quad (-1)^S = (-1)^{\overline{S}} = -1\,, \tag{2.6}$$

that define the usual $N = 4 + 4$ model in type IIA theory.

The model $\langle F, S, \overline{S}, b_1, \overline{b}_1, \overline{b}_2 \rangle$ without the set b_3, defines a theory possessing $N = 2+1$ space-time supersymmetry and SU(3, 1) as a non-compact group of global symmetry [8]. The additional set $\overline{b}_3$ enables us to define the model $\langle F, S, \overline{S}, b_1, \overline{b}_1, \overline{b}_2, \overline{b}_3 \rangle$ in which all left-moving space-time supersymmetries are broken. This results in a $\mathcal{N} = (4, 1)$ superconformal theory on the world-sheet and $N = 2 + 0$ space-time supersymmetry.

Since $\overline{b}_3$ defines a Z_2 action that acts freely, it does not introduce additional states from its twisted sector in the massless spectrum either. Besides, all the bosonic fields from the "parent" $N = 2+1$ theory are projected out apart from the graviton, the dilaton-axion pair and two vector gauge fields. One is the graviphoton of the gravity multiplet and the other corresponds to a matter vector multiplet.

Although the massless spectrum of the $N = 2$ model consists of the gravitational multiplet and just one vector multiplet, it is important to determine the coupling between the vector and scalar fields and the structure of the scalar moduli space. It is well known [8] that the axion-dilaton pair parameterizes a coset space which is topologically a pseudosphere

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\ . \tag{2.7}$$

Let us recall that the $N = 2$ effective supergravity allows a priori two types of couplings between the vector and scalar fields which are distinguished by the *curvature* of the moduli space [8, 5] even though their topological structure is the same. Indeed, when the coupling between the scalar and vector fields is "non-minimal" the Kähler potential is given by

$$K = -3 \log \left(S - \overline{S}\right)\ . \tag{2.8}$$

On the other hand, when the coupling is "minimal" the Kähler potential is given by

$$K = -\log(S - \overline{S})\ . \tag{2.9}$$

From the form of the axion-dilaton kinetic terms in type II superstrings we know that this last case applies to the structure of the axion-dilaton moduli space [8]. The Poincaré half plane is always isometrically embedded in a possibly bigger moduli space G/H.

In the language of $N = 2$ supergravity we recall that S corresponds to a non-homogeneous coordinate on the moduli space [3–5]. Homogeneous coordinates are introduced through $S = \frac{Z}{Z_0}$ and more generally $t_i = \frac{Z_i}{Z_0}$, $i = 1 \ldots n$ if there are $2n+2$ moduli of the vector multiplets. In terms of these, the Kähler potential of vector-moduli space is fixed by the prepotential $F(Z_0, Z_i)$ - which is a holomorphic homogeneous function of degree 2 - through [3–5]

$$K = -\log\ i\mathrm{Im}[\ \overline{Z}_I \partial^I F(Z_J)\ ]\,, \tag{2.10}$$

with the index $I = (0, i)$. It can alternatively be expressed in terms of the non-homogeneous coordinates t_i and the function $f(t_i)$ defined by $F(Z_I) = -iZ_0^2 f(t_i)$ through[2]

$$K = -\log \left[2\left(f\left(t_i\right) + \overline{f}\left(\overline{t}_i\right)\right) - \left(t_i - \overline{t}_i\right)\left(\partial^i f - \overline{\partial}^i \overline{f}\right) \right] . \tag{2.12}$$

In the case of non-minimal coupling with coset space $\mathrm{SU}(1,1)/\,\mathrm{U}(1)$ the prepotential is cubic in Z and given by $F(Z, Z_0) = i\frac{Z^3}{Z_0}$. On the other hand, the prepotential in the case of minimal coupling to the vector fields is $F(Z, Z_0) = Z^2 - Z_0^2$. The latter prepotential is the one we must use to describe the axion-dilaton pair in the case of interest. Therefore, we have shown that it is possible to construct a "minimally" coupled $N = 2$ hyper-free model whose scalar sector forms a special Kähler coset space

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} .$$

This is interesting and rather unexpected. Indeed, in the spirit of the c-map [14] the distinction between the "minimal" and "non-minimal" structures is equivalently expressed by saying that the dimensional reduction to three space-time dimensions of these models on a circle, gives rise to 3d supergravity models with different scalar manifolds (after dualization in three dimensions of the vector fields into scalars). In the case of a "non-minimal" coupling one finds that the scalars together with the scalar-dual of the vector gauge field, parameterize the special quaternionic manifold[3]

$$\frac{G_{2(2)}}{\mathrm{SO}(4)} , \tag{2.13}$$

whereas in the case of "minimal" coupling one finds that the scalar parameterize the special quaternionic-Kähler manifold,

$$\frac{\mathrm{U}(2,2)}{\mathrm{U}(2) \times \mathrm{U}(2)} , \tag{2.14}$$

which is somewhat exotic from the point of view of Calabi-Yau or symmetric orbifold $\mathcal{N} = (2,2)$ compactifications.

3. Hyper-free $N = 2$ theories of the "magic square"

More general hyper-free $N = 2$ models can be constructed with higher number n_V of vector multiplets via asymmetric orbifold constructions starting from type II superstrings with $N = 2 + 4$, $N = 2 + 1$ or eventually $N = 2 + 2$ initial supersymmetry. The breaking of the right-moving supersymmetry via freely acting asymmetric orbifold gives rise to $N = 2 + 0$ hyper-free models. In all constructions of this type, the axion-dilaton pair belongs to one of

[2] In this change of notation one makes use of the invariance of the theory under a Kähler transform of the Kähler potential

$$K \to K + \Lambda + \overline{\Lambda} \tag{2.11}$$

with Λ an arbitrary holomorphic function of the moduli.

[3] One can read [15] for a discussion of special Kähler and special quaternionic manifolds.

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the vector multiplets and it appears always with "minimal coupling". Since in our construction the last projection is assumed to act freely, the scalar Kähler manifold of the final $N = 2$ theory is necessarily a sub-manifold of $\mathcal{M}^{N=2}$ the scalar manifold of the initial "mother" theory with higher supersymmetry, let us consider for instance for N=4+2 or N=2+1:

$$\mathcal{M}^{N=2} \subseteq \frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}, \qquad \text{or} \qquad \mathcal{M}^{N=2} \subseteq \frac{\mathrm{SU}(3,3)}{S(\mathrm{U}(3)\times\mathrm{U}(3))}. \tag{3.1}$$

The special cases where the dimension of the scalar manifold is maximal, $\mathrm{dim}\mathcal{M}_6^{N=2}$=30 and $\mathrm{dim}\mathcal{M}_3^{N=2}$=18, hold for $N = 2$ theories which are known to be in correspondence with the projective special Kähler symmetric spaces of the "magic square" of Freudenthal-Rozenfeld-Tits [2, 1].

These two $N = 2$ "magic" theories are associated to the Jordan algebras $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{C}}$. They are known to have the same scalar field content as some supergravity models with more supersymmetries. The $N = 2$ theory that is associated to $N = 6$ supergravity appears to be the most peculiar one. Not only the scalar fields but also the gauge bosons degrees of freedom are the same.

In the next two sections we provide a construction of the two hyper-free $N = 2$ "magic" theories at the superstring level.

Before that, let us recall some properties of the $N = 6$ supergravity, namely the helicity structure of the $N = 6$ graviton ($\mathbf{G}$) and gravitino ($\mathbf{g}$) supermultiplets

$$\begin{aligned} \mathbf{G} &: \left(+2, +\frac{3}{2}^{6}, +1^{15}, +\frac{1}{2}^{20}, 0^{15}, -\frac{1}{2}^{6}, -1\right) \oplus \left(+1, +\frac{1}{2}^{6}, 0^{15}, -\frac{1}{2}^{20}, -1^{15}, -\frac{3}{2}^{6}, -2\right) \\ \mathbf{g} &: \left(+\frac{3}{2}, +1^{6}, +\frac{1}{2}^{15}, 0^{20}, -\frac{1}{2}^{15}, -1^{6}, -\frac{3}{2}\right) \end{aligned} \tag{3.2}$$

where we indicate the multiplicity of each helicity in exponent. The branching rule under $N = 6 \to N = 2$,

$$\mathbf{G} \to \mathbf{G} \oplus 4\mathbf{g} \oplus 7\mathbf{V} \oplus 4\mathbf{H} \tag{3.3}$$

where $\mathbf{V}$ and $\mathbf{H}$ denote respectively vector-multiplets and hyper-multiplets. The 30 scalars of the $N = 6$ supergravity parameterize a coset

$$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}\ . \tag{3.4}$$

Remarkably, one of the $N = 2$ theories of the magic square [1] possesses exactly the same bosonic content as the $N = 6$ (pure) supergravity. It is associated to the Jordan algebra called $J_3^{\mathbb{H}}$. Both $N = 6$ and $N = 2$ supergravities contain one graviton, 15 vector fields (one of them being the graviphoton and 14 belonging to vector multiplets in the case of the $N = 2$) and the 30 scalars with the structure we have just indicated.

Before we proceed further, we would like to make some remarks concerning the fermionic spectrum: from the decomposition of the eq.(3.3), $N = 6$ supergravity contains 26 extra fermions in addition to the 2 gravitini from the graviton multiplet $G|_{N=2}$, namely 22 have spin 1/2 and the last 4 are spin-3/2 gravitini. This is the same as the number of fermionic degrees of freedom of the $J_3^{\mathbb{H}}$ supergravity. Hence the difference is that the extra 4 gravitini must be replaced in the $N = 2$ model by 4 spin-1/2 fermions.

4. Superstring construction of the magic $\mathcal{M}_6^{N=2}$

This section is devoted to the superstring construction of the exceptional $N = 2$ supergravity based on the Jordan algebra $J_3^{\mathbb{H}}$ [1] which contains at the massless level the same number of scalars and gauge bosons as pure $N = 6$ supergravity.

The simplest way to get a $N = 6$ theory in the type IIA setup is to start from the type II superstring with $N = 4 + 4$ supersymmetry and then to use a freely acting asymmetric Z_2 orbifold which reduces the left-moving supersymmetries to $N = 2 + 4$ [8, 10, 11]. In the fermionic construction language [6], we start with the $N = 4 + 4$ model $\langle F, S, \overline{S}\rangle$. Then, the Z_2 asymmetric breaking to $N = 2 + 4$ is obtained by choosing one additional supersymmetry breaking set for instance b' [8],

$$b' = \left\{\psi^\mu, \chi^{1,2}, y^{3,4,5,6}, y^1, w^1 \,|\, \overline{y}^1, \overline{w}^1\right\}, \tag{4.1}$$

and fixing the GGSO projection by the choice of sign $(-1)^{b'} = -1$.
Although the $\langle F, S, \overline{S}, b'\rangle$ model defines an initial $N = 2 + 4$ theory, it will turn out that it is not a good starting point to reduce the right-moving gravitini and obtain the magic $N = 2 + 0$ theory.

4.1 A failed attempt

Indeed, we could add an additional breaking set $\overline{b}_{\overline{S}}$

$$\overline{b}_{\overline{S}} = \left\{y^2, w^2 \,|\, \overline{y}^2, \overline{w}^2\right\} \cup \overline{S} \tag{4.2}$$

that would break $N = 2{+}4$ to $N = 2{+}0$ removing the four right- moving gravitini. However, this would remove at the same time the RR-scalars and so, the number of remaining scalar degrees of freedom of the $N = 2$ theory would be 14 instead of 30. The scalar manifold of this hyper-free $N = 2$ model, $\langle F, S, \overline{S}, b', \overline{b}_{\overline{S}}\rangle$,

$$\mathcal{M}^{N=2} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,6)}{\mathrm{SO}(2)\times\mathrm{SO}(6)} \;\subset\; \frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)} \tag{4.3}$$

is a sub-manifold of the desired $\mathcal{M}_6^{N=2}$ with dimension 14 instead of 30.

To obtain the desired "magic" $N = 2 + 0$ it seems necessary to construct in a different way the initial $N = 2 + 4$ theory so that, the RR scalars and gauge bosons survive the additional supersymmetry breaking projection from $N = 2 + 4$ to $N = 2 + 0$.

4.2 $N = (2 + 2) + 4$ construction with $(3/2) \leftrightarrow (1/2)$ magic twist

To define the asymmetric orbifold mechanism which has the property to replace the 4 gravitini of the $N = 6$ by 4 fermions and at the same time keeps the RR-scalars, it is necessary to prepare the initial $N = 2 + 4$ theory in a more sophisticated way that we will describe in detail below.

We start with the string model $\langle F, S, \overline{S}, b\rangle$, where b is *purely left-moving*

$$b = \left\{\psi^\mu, \chi^{1,2}, y^{3,4,5,6}\right\}. \tag{4.4}$$

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b defines an asymmetric, non-freely acting orbifold acting on $\langle F, S, \overline{S}\rangle$ model and breaks the supersymmetry (in the untwisted sector), from $N = 4{+}4 \to N = 2{+}4$. The twisting action of b on the untwisted sector is thus similar to the one of b'. There is however a *fundamental difference* between b and b'; in b all elements are left-moving. This holomorphic property will be crucial in what follows.

Different choices of GGSO projection give different models. Choosing for instance $(-1)^b = -1$, the massless spectrum would be given by:

$\hat{\emptyset}$ *super-sector*

$$\left[\left(\psi^\mu,\chi^{1,2}\right)\,\oplus\, sp\left(\psi^\mu,\chi^{1,2}\right)_-\, sp\left(\chi^{3,4,5,6}\right)_+\right]\left[\left(\overline{\psi}^\mu,\overline{\chi}^{1,\dots,6}\right)\,\oplus\, sp\left(\overline{\psi}^\mu,\overline{\chi}^{1,\dots,6}\right)_-\right], \tag{4.5}$$

$\hat{b}$ *super-sector*

$$\left[sp\left(\psi^\mu,\chi^{1,2}\right)_- sp\left(y^{3,4,5,6}\right)_+ \oplus sp\left(\chi^{3,4,5,6}\right)_- sp\left(y^{3,4,5,6}\right)_+\right]\otimes\left[\left(\overline{\psi}^\mu,\overline{\chi}^{1,\dots,6}\right)\oplus sp\left(\overline{\psi}^\mu,\overline{\chi}^{1,\dots,6}\right)_-\right]. \tag{4.6}$$

In each super-sector our notation indicates the massless states in the sectors $[NS \oplus R] \otimes [\overline{NS} \oplus \overline{R}]$. Even though the untwisted $\hat{\emptyset}$ super-sector corresponds to a theory with $N = 2{+}4$ supersymmetry as in the previous b'-construction, here the supersymmetry is extended to $N = (2+2)+4$ because now the twisted super-sector $\hat{b}$ contains massless states and among them *two extra left-moving gravitini*. This supersymmetric extensions happens due to the left-moving "holomorphic structure" of b. Thus, the $\langle F, S, \overline{S}, b\rangle$ twisted construction still has maximal $N = 8$ supersymmetry constructed in a b-twisted manner. We should stress here that this supersymmetric extension from $N = 6 \to N = 8$ of the $\langle F, S, \overline{S}, b\rangle$ model *is a stringy phenomenon* and can be seen algebraically at the level of the modular invariant partition function which implies the inclusion in the spectrum of the b-twisted sectors with $h \neq 0$:

$$\begin{aligned} Z_b = &\frac{1}{|\eta|^8}\frac{1}{4}\sum_{h,g} Z_{6,6}\left[{}^h_g\right]|_{\mathrm{SO}(6)} \sum_{a,b}(-1)^{a+b+ab}\,\theta\begin{bmatrix}a+h\\ b+g\end{bmatrix}\theta\begin{bmatrix}a-h\\ b-g\end{bmatrix}\theta\begin{bmatrix}a\\ b\end{bmatrix}^2 \\ &\times\frac{1}{2}\sum_{\overline{a},\overline{b}}(-1)^{\overline{a}+\overline{b}+\overline{a}\overline{b}}\,\overline{\theta}\begin{bmatrix}\overline{a}\\ \overline{b}\end{bmatrix}^4 \end{aligned} \tag{4.7}$$

where $Z_{6,6}\left[{}^h_g\right]\Big|_{\mathrm{SO}(6)}$ is the contribution of the six internal coordinates $\partial\phi^I_L = (y\omega)^I$, $I = 1, 2, \dots 6$. The directions $\phi_L^{3,4,5,6}$ are twisted by Z_2 induced by b. Remember that in the fermionic construction the coordinate currents are given in terms of the two dimensional free fermions, so that the Z_2 acts on $y^{3,4,5,6}$ only. $\omega^{3,4,5,6}$, $y^{1,2}$ and $\omega^{1,2}$ are invariant under Z_2.

$$\begin{aligned} Z_{6,6}\begin{bmatrix}h\\ g\end{bmatrix}_{\mathrm{SO}(6)} = &\frac{1}{2|\eta|^4}\sum_{\gamma,\delta}\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}^2_{y^{1,2},\omega^{1,2}}\ \overline{\theta}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}^2_{\overline{y}^{1,2},\overline{\omega}^{1,2}} \\ &\times\frac{(-1)^{\gamma g+\delta h}}{|\eta|^8}\,\theta\begin{bmatrix}\gamma\\ \delta\end{bmatrix}^2_{\omega^{3,4,5,6}}\theta\begin{bmatrix}\gamma+h\\ \delta+g\end{bmatrix}_{y^{3,4}}\theta\begin{bmatrix}\gamma-h\\ \delta-g\end{bmatrix}_{y^{5,6}}\overline{\theta}\begin{bmatrix}\gamma\\ \delta\end{bmatrix}^4_{\overline{y}^{3,4,5,6},\overline{\omega}^{3,4,5,6}}. \end{aligned} \tag{4.8}$$

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To see that this leads to an $N = 4 + 4$ supersymmetry we first perform the sum over the (a, b) indices using the Jacobi identity

$$\frac{1}{2}\sum_{a,b}(-1)^{a+b+ab}\,\theta\begin{bmatrix}a+h\\b+g\end{bmatrix}(v)\,\theta\begin{bmatrix}a-h\\b-g\end{bmatrix}(v)\,\theta\begin{bmatrix}a\\b\end{bmatrix}^2(v) = -\theta\begin{bmatrix}1\\1\end{bmatrix}^2(v)\,\theta\begin{bmatrix}1+h\\1+g\end{bmatrix}(v)\,\theta\begin{bmatrix}1-h\\1-g\end{bmatrix}(v). \tag{4.9}$$

This partial summation shows that the partition function has a second order zero for $v \to 0$ from the left-moving sector. This can be traced to the presence of two left-moving massless gravitini in the untwisted sector and indicates *at least* $N = 2$ space-time supersymmetry from this sector. However to see that the *full* $N = 4 + 4$ supersymmetry is present we need to show that an extra double zero is present in the partition function. Then we are left to compute:

$$\mathcal{S} = \frac{1}{2}\sum_{h,g}(-1)^{\gamma g+\delta h}\theta\begin{bmatrix}1+h\\1+g\end{bmatrix}\,\theta\begin{bmatrix}1-h\\1-g\end{bmatrix}\,\theta\begin{bmatrix}\gamma+h\\\delta+g\end{bmatrix}\,\theta\begin{bmatrix}\gamma-h\\\delta-g\end{bmatrix}. \tag{4.10}$$

Defining

$$(A, B) = (1-h, 1-g); \qquad (\gamma, \delta) = (1+H, 1+G)\,, \tag{4.11}$$

and using the Jacobi identity associated to (A, B), one can show that:

$$\begin{aligned}\mathcal{S} &= (-1)^{GH+G}\,\frac{1}{2}\sum_{A,B}(-1)^{A+B}\,\theta\begin{bmatrix}A\\B\end{bmatrix}^2(v)\,\theta\begin{bmatrix}A+H\\B+G\end{bmatrix}(v)\,\theta\begin{bmatrix}A-H\\B-G\end{bmatrix}(v)\\ &= (-1)^{G(H+1)}\,\theta\begin{bmatrix}1\\1\end{bmatrix}^2(v)\,\theta\begin{bmatrix}1+H\\1+G\end{bmatrix}(v)\,\theta\begin{bmatrix}1-H\\1-G\end{bmatrix}(v)\\ &= \theta\begin{bmatrix}1\\1\end{bmatrix}^2(v)\,\theta\begin{bmatrix}\gamma\\\delta\end{bmatrix}^2(v),\end{aligned} \tag{4.12}$$

so that overall

$$Z_b = \frac{1}{|\eta|^8}\,\theta\begin{bmatrix}1\\1\end{bmatrix}^4\,\bar\theta\begin{bmatrix}1\\1\end{bmatrix}^4\,\frac{1}{2}\sum_{\gamma,\delta}\,\theta\begin{bmatrix}\gamma\\\delta\end{bmatrix}^6\,\bar\theta\begin{bmatrix}\gamma\\\delta\end{bmatrix}^6\,. \tag{4.13}$$

Thus, we have shown explicitly that the $b-$twisted partition function exhibits a zero of order four on the holomorphic and anti-holomorphic sectors. The two extra zero's correspond to the presence of two left-moving massless gravitini in the $b-$twisted super-sector as we have mentioned above. Actually the computation of the string helicity super-traces [16, 17] would let us conclude that the $\langle F, S, \overline{S}, b\rangle$ model has a maximal $N = 8$ supersymmetry. Other choices of the GGSO projection coefficients define non trivial lattice shifts. Choosing for instance the "factorized point" of the $Z_{6,6}$ such that

$$Z_{6,6}\begin{bmatrix}h\\g\end{bmatrix}_{\rm twisted} = Z_{4,4}\begin{bmatrix}h\\g\end{bmatrix}_{\rm twisted}\,\frac{\Gamma_{2,2}(T,U)}{|\eta|^4} \tag{4.14}$$

where the twisted lattice $Z_{4,4}\left[{h \atop g}\right]_{\rm twisted}$ correspond to the contribution of the $\partial\phi^I = y^I\omega^I$, $I = 3, 4, 5, 6$ directions which are twisted by Z_2. The $\Gamma_{2,2}(T, U)$ lattice is the contribution of the untwisted directions $\partial\phi^{1,2} = (y\omega)^{1,2}$. The latter depends on the T, U moduli which are associated to the 2-torus. In the fermionic construction T and U are fixed to the self-dual point $T = U = i$. A way to break the right-moving space-time supersymmetry "spontaneously" is to correlate the right-moving helicity with a lattice shift. However, in order to preserve the bosonic content of the massless spectrum it is necessary to keep some of the twisted Ramond-Ramond states. This leads us to correlate the helicity characters $(\overline{a}, \overline{b})$, the twisted (h, g) characters with a lattice shift of the $\Gamma_{2,2}$ lattice [18].

To do that one defines the shifted lattice sum as

$$\Gamma_{2,2}\begin{bmatrix}\overline{a}+h\\ \overline{b}+g\end{bmatrix} = \sum_{n_i,m_j} (-1)^{(\overline{a}+h)m_1+(\overline{b}+g)n_1+m_1n_1} \times\frac{T_2}{\tau_2}\exp\left[-2i\pi B_{ij}m^in^j - \pi G_{ij}\frac{(m^i+n^i\tau)(m^j+n^j\overline{\tau})}{\operatorname{Im}\tau}\right] \tag{4.15}$$

with

$$G_{ij} = \frac{\operatorname{Im}T}{\operatorname{Im}U}\begin{bmatrix}1 & \operatorname{Re}U\\ \operatorname{Re}U & |U|^2\end{bmatrix}, \qquad B_{ij} = \epsilon_{ij}\operatorname{Re}T \tag{4.16}$$

written in the Poisson dual form. This lattice sum differs from $\Gamma_{2,2}(T, U)$ by the introduction of the modular invariant phase $(-1)^{(\overline{a}+h)m_1+(\overline{b}+g)n_1+m_1n_1}$. Note that the right helicity shift has the usual form of a right-moving "temperature" coupling [18] while the $(-1)^{hm_1+gn_1}$ shift acts on the twisted sectors (h, g) of the Z_2 orbifold. This phase modification does not change the modular covariance properties of the lattice sum. Its effect is to make massive the sectors corresponding to $\overline{a} + h = 1$ *mod* 2.

Some comments are in order:

- In the $\hat{\emptyset}$ super-sector we loose the R-R and NS-R sectors which contain 8 vectors, 16 scalars, 12 spin 1/2 fermions and 4 gravitini coming from the right moving side.
- The $\hat{b}$ super-sector contains the NS-R and R-R sectors which provide us with 8 vectors, 16 scalars and 16 spin 1/2 fermions.

We see that overall, all these operations have had no effect on the bosonic content of the theory with respect to the $N = 2 + 4$ model corresponding to the b'-orbifold. However, from the fermionic fields point of view we have lost 4 gravitini but gained 4 additional spin 1/2 fermions. This is what we call a $(3/2) \leftrightarrow (1/2)$ twist.

Given the number of vector fields it is clear that no hypermultiplet is present in the spectrum. However since 16 among the 30 scalar fields now come from the b-twisted sector it is not immediate to conclude what is the special Kähler manifold that is associated to the scalars. Several possibilities exist with this dimension namely

$$\frac{\mathrm{SU}(1,15)}{\mathrm{U}(1)\times\mathrm{SU}(15)}; \qquad \frac{\mathrm{SO}(2,14)}{\mathrm{SO}(2)\times\mathrm{SO}(14)}\times\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}; \qquad \frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)} \tag{4.17}$$

The first two need either a rank 15 or rank 9 symmetry group to be realized in a linear way which is a too large symmetry to be realized in the model under consideration. On the other hand, one can realize explicitly a rank 6 symmetry group through

$$\mathrm{SO}(2)_{\chi^{1,2}} \times \mathrm{U}(1)_{\phi,a} \times \mathrm{SU}(2)^+_{y^{3,4,5,6}} \times \mathrm{SU}(4)_{\overline{\chi}^{3,4,5,6}} \subset \mathrm{U}(6) \tag{4.18}$$

where, in the above equation, the fields associated to each group factor are indicated as lower indices.

Furthermore, from the coset decomposition of

$$\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)} \quad\rightarrow\quad \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,6)}{\mathrm{SO}(2)\times\mathrm{SO}(6)}\right)_{h=0} \times \left(\frac{\mathrm{SU}(2,4)}{\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(4)}\right)_{h=1}, \quad (4.19)$$

we recognize the coset structure of the 14 untwisted moduli from the $\hat{\emptyset}$ super-sector ($h=0$) and the 16 RR-moduli coming from the twisted $\hat{b}$-super-sector ($h=1$). This decomposition is identical to the one of $N=2+6$ model described in ref. [8].

We shall momentarily proceed to the construction of the other four dimensional magic models, but we would like to make some comments on the reduction and oxidation of $\mathcal{M}_6^{N=2}$ to three and five space-time dimensions:

(i) The three dimensional case is obtained via S^1 compactification. The sets $S,\overline{S}$ and b are taken to be the same as in the four dimensional construction. In three dimensions however the dimension of the scalar manifold is extended via 3d duality transformation of the vector gauge bosons to scalars. In the untwisted $h=0$ sector the dimension of the 3d scalar manifold becomes:
14 (4d-scalars)+**14** (4d-vectors)+**2** (4d-graviphoton)+**2** (3d-graviphoton, $g_{3,\mu}$)= **32**.
Altogether parameterize (via the $c-$map) the quaternionic manifold,

$$\mathcal{M}_{h=0}^{D=3} = \frac{\mathrm{SO}(4,8)}{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(8)} \quad (4.20)$$

In the $h=1$ sector the dimension of the 3d scalar manifold becomes:
16 (4d-RR scalars) + **16** (4d-RR vectors) = **32**.
The 32 scalars from the $h=0$ sector together with the 32 scalars from $h=1$ sector parameterize in 3d the quaternionic manifold of the magic square [2, 1]:

$$\mathcal{M}_6^{D=3} = \frac{E_{7(-5)}}{\mathrm{SU}(2)\times\mathrm{SO}(12)}, \quad (4.21)$$

as expected by a $c-$map operation on the four dimensional $\mathcal{M}_6^{N=2}$ magic model.

(ii) The five dimensional case is obtained from $\mathcal{M}_6^{N=2}$ magic model via one dimensional oxidation. Here also the sets $S,\overline{S}$ and b are the same as in the four dimensional case. The only difference is the replacement of the two dimensional lattice $\Gamma_{2,2}$ by the one dimensional lattice $\Gamma_{1,1}$. Also, we identify $\chi^1\equiv\psi^5$ and $y^1w^1\equiv\partial X^5$, $\overline{y}^1\overline{w}^1\equiv\overline{\partial}X^5$; X^5 is taken non-compact. In five dimensions the number of scalars is reduced since they are becoming the 5th components of higher spin fields. In the $h=0$ sector the 6 scalars parameterize the manifold

$$\mathcal{M}_{h=0}^{D=5} = \mathrm{SO}(1,1)\times\frac{\mathrm{SO}(1,5)}{\mathrm{SO}(5)} \equiv \mathrm{SO}(1,1)\times\frac{\mathrm{SO}(1,5)}{\mathrm{SP}(4)} \quad (4.22)$$

In the $h=1$ sector the RR-states are decomposed in 5d vectors and 5d scalars. The $\frac{1}{2}$ of the 4d-scalar degrees of freedom in this sector are eaten by the 5d-vectors. So we are left with 8 5d-scalars that parameterize the manifold

$$\mathcal{M}^{D=5}_{h=1} = \frac{\mathrm{SP}(2,4)}{\mathrm{SP}(2)\times\mathrm{SP}(4)} \tag{4.23}$$

The 6 scalars from $h = 0$ sector together with the 8 from the $h = 1$ sector parameterize in five space-time dimensions the manifold of the magic square [2, 1],

$$\mathcal{M}^{D=5}_{6} = \frac{\mathrm{SU}^*(6)}{\mathrm{USP}(6)}\,, \tag{4.24}$$

as expected by the supersymmetry conserving operations via *Oxidation* ↔ *Reduction*.

5. Superstring construction of the magic $\mathcal{M}^{N=2}_3$

The other $N = 2$ "magic" theory we would like to construct in four space-time dimensions is the hyper-free theory which has the bosonic spectrum of the $N = 3$ supergravity coupled to three extra vector multiplets. The scalar manifold is Kähler and contains 18 scalars.

$$\mathcal{M}_3 = \frac{\mathrm{SU}(3,3)}{S(\mathrm{U}(3)\times\mathrm{U}(3))} \tag{5.1}$$

Here also the string construction has to be asymmetric involving asymmetric twists and lattice shifts.

Our starting point is a twisted realization of the $N = 8$ based on the holomorphic and anti-holomorphic basis sets

$$\begin{aligned}
b'_1 &= \left\{\psi^\mu, \chi^{1,2}, y^{3,4}, y^{5,6}\right\}\\
\overline{b}'_1 &= \left\{\overline{\psi}^\mu, \overline{\chi}^{1,2}, \overline{y}^{3,4}, \overline{y}^{5,6}\right\}\\
\overline{b}'_2 &= \left\{\overline{\psi}^\mu, \overline{\chi}^{3,4}, \overline{y}^{1,2}, \overline{y}^{5,6}\right\}.
\end{aligned} \tag{5.2}$$

The holomorphic set A

$$A = \left\{y^{3,4}y^{5,6}w^{3,4}w^{5,6}\ \right\} \tag{5.3}$$

will be used in our construction as well. b'_1 induces a $Z^2_{b'_1}$ projection which seems to break the left-moving supersymmetry from 4 to 2. Also $\overline{b}'_1, \overline{b}'_2$ induce $Z^2_{\overline{b}'_1} \times Z^2_{\overline{b}'_2}$ projections that seem to break the right-moving supersymmetry from 4 to 1. However, these supersymmetry breakings are not efficient in general due to the (anti-) holomorphic structure of the basis sets that imply the presence of extra gravitini in the twisted sectors of the theory. Thus, there is a choice of the GGSO coefficients where the supersymmetry is maximal. Explicitly, this choice defines the following partition function for the $N = (2 + 2) + (1 + 1 + 1 + 1)$

model $\{S, b_1'; \overline{S}, \overline{b}_1', \overline{b}_2'; A\}$

$$
\begin{aligned}
Z_{N=8} = \frac{1}{|\eta|^{24}} \; & \frac{1}{2}\sum_{h_1,g_1,} \; \frac{1}{2}\sum_{\overline{h}_1,\overline{g}_1,} \; \frac{1}{2}\sum_{\overline{h}_2,\overline{g}_2} \; \frac{1}{2}\sum_{\gamma,\delta} \; \frac{1}{2}\sum_{A,B} \\
& \times \; \frac{1}{2}\sum_{a,b}(-1)^{a+b+ab} \; \theta\begin{bmatrix} a \\ b \end{bmatrix} \; \theta\begin{bmatrix} a \\ b \end{bmatrix} \; \theta\begin{bmatrix} a+h_1 \\ b+g_1 \end{bmatrix} \; \theta\begin{bmatrix} a-h_1 \\ b-g_1 \end{bmatrix} \; (-)^{h_1 g_1 + AB} \\
& \times \; \theta\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \theta\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \; \theta\begin{bmatrix} \gamma-A \\ \delta-B \end{bmatrix} \theta\begin{bmatrix} \gamma+A+h_1 \\ \delta+B+g_1 \end{bmatrix} \; \theta\begin{bmatrix} \gamma-A \\ \delta-B \end{bmatrix} \theta\begin{bmatrix} \gamma+A-h_1 \\ \delta+B-g_1 \end{bmatrix} \\
& \times \; \overline{\theta}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \overline{\theta}\begin{bmatrix} \gamma+\overline{h}_1 \\ \delta+\overline{g}_1 \end{bmatrix} \; \overline{\theta}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \overline{\theta}\begin{bmatrix} \gamma-\overline{h}_2 \\ \delta-\overline{g}_2 \end{bmatrix} \; \overline{\theta}\begin{bmatrix} \gamma \\ \delta \end{bmatrix} \overline{\theta}\begin{bmatrix} \gamma-\overline{h}_1+\overline{h}_2 \\ \delta-\overline{g}_1+\overline{g}_2 \end{bmatrix} \; (-)^{\overline{h}_1\overline{g}_1+\overline{h}_2\overline{g}_2} \\
& \times \; \frac{1}{2}\sum_{\overline{a},\overline{b}}(-1)^{\overline{a}+\overline{b}+\overline{a}\overline{b}} \; \overline{\theta}\begin{bmatrix} \overline{a} \\ \overline{b} \end{bmatrix} \; \overline{\theta}\begin{bmatrix} \overline{a}+\overline{h}_1 \\ \overline{b}+\overline{g}_1 \end{bmatrix} \; \overline{\theta}\begin{bmatrix} \overline{a}+\overline{h}_2 \\ \overline{b}+\overline{g}_2 \end{bmatrix} \; \overline{\theta}\begin{bmatrix} \overline{a}-\overline{h}_1-\overline{h}_2 \\ \overline{b}-\overline{g}_1-\overline{g}_2 \end{bmatrix} \quad (5.4)
\end{aligned}
$$

The choice of the phases and the arguments of the θ-functions is dictated by modular invariance and the existence of maximal supersymmetry. Indeed, the existence of 4 left-moving supersymmetries can be shown explicitly by using the Jacobi identity associated to the the arguments (a, b) and then to (h_1, g_1) as previously. The 4 right-moving supersymmetries can be visualized by using first the Jacobi identity associated to $(\overline{a}, \overline{b})$ and then the one associated to $(\overline{h}_1, \overline{g}_1)$ and then to $(\overline{h}_2, \overline{g}_2)$.

The massless spectrum of the above $N = 8$-twisted construction is the following:

$\hat{\emptyset}$ *super-sector*

$$
\begin{aligned}
&\left[(\psi^\mu, \chi^{1,2}) \,\oplus\, sp(\psi^\mu, \chi^{1,2})_- sp(\chi^{3,4,5,6})_+\right] \qquad (5.5)\\
&\qquad \otimes \left[\overline{\psi}^\mu \,\oplus\, sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{1,2})_- sp(\overline{\chi}^{3,4})_+ sp(\overline{\chi}^{5,6})_+\right],
\end{aligned}
$$

$\hat{b}_1'$ *super-sector*

$$
\begin{aligned}
&\left[(sp(\psi^\mu, \chi^{1,2})_- sp(y^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(y^{3,4,5,6})_+\right] \qquad (5.6)\\
&\qquad \otimes \left[(\overline{\psi}^\mu, \overline{\chi}^{1,2}) \,\oplus\, sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{1,2})_- sp(\overline{\chi}^{3,4})_+ sp(\overline{\chi}^{5,6})_+\right],
\end{aligned}
$$

$\hat{\overline{b}}_1'$ *super-sector*

$$
\begin{aligned}
&\left[(\psi^\mu, \chi^{1,2}) \,\oplus\, sp(\psi^\mu, \chi^{1,2})_- sp(\chi^{3,4,5,6})_+\right] \qquad (5.7)\\
&\qquad \otimes \left[\left(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{1,2})_+ \oplus sp(\overline{\chi}^{3,4})_+ sp(\overline{\chi}^{5,6})_+\right) \oplus sp(\overline{y}^{3,4})_+ sp(\overline{y}^{5,6})_+\right],
\end{aligned}
$$

$\hat{\overline{b}}_2'$ *super-sector*

$$
\begin{aligned}
&\left[(\psi^\mu, \chi^{1,2}) \,\oplus\, sp(\psi^\mu, \chi^{1,2})_- sp(\chi^{3,4,5,6})_+\right] \qquad (5.8)\\
&\qquad \otimes \left[\left(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{3,4})_+ \oplus sp(\overline{\chi}^{1,2})_+ sp(\overline{\chi}^{5,6})_+\right) \oplus sp(\overline{y}^{1,2})_+ sp(\overline{y}^{5,6})_+\right],
\end{aligned}
$$

$\hat{\overline{b}}_1'\hat{\overline{b}}_2'$ *super-sector*

$$
\begin{aligned}
&\left[(\psi^\mu, \chi^{1,2}) \,\oplus\, sp(\psi^\mu, \chi^{1,2})_- sp(\chi^{3,4,5,6})_+\right] \qquad (5.9)\\
&\qquad \otimes \left[\left(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{5,6})_+ \,\oplus\, sp(\overline{\chi}^{1,2})_+ sp(\overline{\chi}^{3,4})_+\right) \oplus sp(\overline{y}^{1,2})_+ sp(\overline{y}^{3,4})_+\right],
\end{aligned}
$$

$\hat{b}'_1\hat{\overline{b}}{}'_1$ *super-sector*

$$\begin{aligned}&\left[(sp(\psi^\mu,\chi^{1,2})_- sp(y^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(y^{3,4,5,6})_+\right] \qquad (5.10)\\ &\qquad\otimes\left[\left(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{1,2})_+ \,\oplus\, sp(\overline{\chi}^{3,4})_+ sp(\overline{\chi}^{5,6})_+\right) \oplus sp(\overline{y}^{3,4})_+ sp(\overline{y}^{5,6})_+\right],\end{aligned}$$

$\hat{b}'_1\hat{\overline{b}}{}'_2$ *super-sector*

$$\begin{aligned}&\left[(sp(\psi^\mu,\chi^{1,2})_- sp(y^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(y^{3,4,5,6})_+\right] \qquad (5.11)\\ &\qquad\otimes\left[\left(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{3,4})_+ \,\oplus\, sp(\overline{\chi}^{1,2})_+ sp(\overline{\chi}^{5,6})_+\right) \oplus sp(\overline{y}^{1,2})_+ sp(\overline{y}^{5,6})_+\right],\end{aligned}$$

$\hat{b}'_1\hat{\overline{b}}{}'_1\hat{\overline{b}}{}'_2$ *super-sector*

$$\begin{aligned}&\left[(sp(\psi^\mu,\chi^{1,2})_- sp(y^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(y^{3,4,5,6})_+\right] \qquad (5.12)\\ &\qquad\otimes\left[\left(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{5,6})_+ \,\oplus\, sp(\overline{\chi}^{1,2})_+ sp(\overline{\chi}^{3,4})_+\right) \oplus sp(\overline{y}^{1,2})_+ sp(\overline{y}^{3,4})_+\right],\end{aligned}$$

Even though the untwisted $\hat{\emptyset}$ super-sector corresponds to a theory with $N = 2 + 1$ supersymmetry, the supersymmetry is extended to $N = (2 + 2) + (1 + 3)$ because of the extra left- and right-moving gravitini arising from the $\hat{b}'_1$, $\hat{\overline{b}}{}'_1$,$\hat{\overline{b}}{}'_2$ and $\hat{\overline{b}}{}'_1\hat{\overline{b}}{}'_2$ twisted super-sectors. There are two extra left-moving gravitini from $\hat{\overline{b}}{}'_1$ and three right moving ones from $\hat{\overline{b}}{}'_1$,$\hat{\overline{b}}{}'_2$ and $\hat{\overline{b}}{}'_1\hat{\overline{b}}{}'_2$. This supersymmetric extensions happens due to the left-and right- moving "holomorphic" structure of $b'_1, \overline{b}'_2, \overline{b}'_3$. Thus, the twisted construction still has maximal $N = 8$ supersymmetry constructed in a twisted manner.

A way to reduce the left- and right- supersymmetry is to couple the lattice characters (γ, δ) and (A, B) to the left- and right- helicities and (twisted) R-symmetry charges. We will first construct two versions of the $N = 2 + (1 + 1)$ supergravity model containing six and eight extra vector multiplets. Then we will reduce further the supersymmetry to obtain the magic $\mathcal{M}_3^{N=2}$.

5.1 $N = 2 + (1 + 1)$ supergravity, with $\mathcal{M}^{N=4}_{n_A} = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,\ n_A)}{S(6)\times \mathrm{SO}(n_A)}$

In order to reduce the supersymmetry to $N = 2 + (1 + 1)$ one has to eliminate from the massless sectors the gravitini coming from the $\hat{b}'_1$, $\hat{\overline{b}}{}'_1$ and $\hat{\overline{b}}{}'_2$ twisted super-sectors. One way to do that is to use the lattice characters (γ, δ) and (A, B) to impose Z_2 projections. Inserting in the $N = 8$ partition function $Z_{N=8}$ the phase

$$\begin{aligned}Z_{N=8} \quad\longrightarrow\quad & Z^A_{N=4} \qquad (5.13)\\ &= Z_{N=8}\ (-)^{\delta(A+h_1+\overline{h}_1+\overline{h}_2)+\gamma(B+g_1+\overline{g}_1+\overline{g}_2)+(A+h_1+\overline{h}_1+\overline{h}_2)(B+g_1+\overline{g}_1+\overline{g}_2)}\end{aligned}$$

imposes in the massless states the constraint

$$(-)^{A+h_1+\overline{h}_1+\overline{h}_2} = +1 \qquad (5.14)$$

which eliminates the $\hat{b}'_1$, $\hat{\overline{b}}{}'_1$, $\hat{\overline{b}}{}'_2$ as well as the $\hat{b}'_1\hat{\overline{b}}{}'_1\hat{\overline{b}}{}'_2$ super-sectors. Naively one obtains a $N = 2 + 2$ supergravity model with two left- one right-moving supersymmetries from the $\hat{\emptyset}$ super-sector and one right-moving supersymmetry from the $\hat{\overline{b}}{}'_1\hat{\overline{b}}{}'_2$ super-sector.

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The same sectors can be eliminated with a different choice of the phase,

$$
\begin{aligned}
Z_{N=8} \quad \longrightarrow \quad & Z_{N=4} \\
& = Z_{N=8}\ (-)^{\delta(h_1+\overline{h}_1+\overline{h}_2)+\gamma(g_1+\overline{g}_1+\overline{g}_2)+(h_1+\overline{h}_1+\overline{h}_2)(g_1+\overline{g}_1+\overline{g}_2)}
\end{aligned}
\tag{5.15}
$$

imposing the constraint

$$
(-)^{h_1+\overline{h}_1+\overline{h}_2} = +1 \tag{5.16}
$$

Both $Z^A_{N=4}$ and $Z_{N=4}$ have $N = 4$ supersymmetry. However, due to the holomorphic structure of the A-set in the $Z^A_{N=4}$ model, some extra massless states arise from the $\hat{A}\hat{b}'_1\hat{\overline{b}}'_1\hat{\overline{b}}'_2$ super-sector. In the $Z_{N=4}$ as well as in the initial $Z_{N=8}$ model these extra massless states are projected out due to the B-projection.

The massless states of the $Z_{N=4}$ are those of the $N = 4$ supergravity coupled to six extra vector multiplets. There are 38 scalars parameterizing the manifold:

$$
\mathcal{M}^{N=4}_6 = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6)\times\mathrm{SO}(6)}\ . \tag{5.17}
$$

In the $Z^A_{N=4}$ construction there are extra massless states from the $\hat{A}\hat{b}'_1\hat{\overline{b}}'_1\hat{\overline{b}}'_2$ super-sector $\hat{A}\hat{b}'_1\hat{\overline{b}}'_1\hat{\overline{b}}'_2$ *super-sector*

$$
\begin{aligned}
&\Big[(sp\left(\psi^\mu,\chi^{1,2}\right)_-\ sp\left(w^{3,4,5,6}\right)_+\ \oplus\ sp\left(\chi^{3,4,5,6}\right)_+\ sp\left(w^{3,4,5,6}\right)_+\Big] \\
&\qquad\qquad \otimes \Big[\Big(sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{5,6})_+\ \oplus\ sp(\overline{\chi}^{1,2})_+ sp(\overline{\chi}^{3,4})_+\Big) \oplus sp(\overline{y}^{1,2})_+ sp(\overline{y}^{3,4})_+\Big]\ .
\end{aligned}
\tag{5.18}
$$

In the $Z^A_{N=4}$ the extra vector multiplets are eight and the total number of scalars is 50 that parameterize the manifold:

$$
\mathcal{M}^{N=4}_8 = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,8)}{\mathrm{SO}(6)\times\mathrm{SO}(8)}\ . \tag{5.19}
$$

In the next subsection starting from the $\mathcal{M}^{N=4}_8$ model, we will construct the magic $\mathcal{M}^{N=2}_3$ by an "helicity twisting mechanism" similar to the one we had introduced in the construction of the $\mathcal{M}^{N=2}_6$.

5.2 $\mathcal{M}^{N=4}_8 \longrightarrow$ magic $\mathcal{M}^{N=2}_3$

The magic $\mathcal{M}^{N=2}_3$ can be obtained from the asymmetric $Z^A_{N=4}$ construction breaking the 2 right-moving supersymmetry via the insertion of a phase

$$
\begin{aligned}
Z^A_{N=4} \ \longrightarrow\ & Z^{\mathrm{Magic}_3}_{N=2} \\
& = \ Z^A_{N=4}\ (-)^{B(h_1+\overline{a})+A(g_1+\overline{b})+AB}
\end{aligned}
\tag{5.20}
$$

which couples the left-twisted arguments (h_1, g_1) and the right-helicity charges $(\overline{a}, \overline{b})$ with the lattice arguments (A, B). The phase factor $(-)^{AB}$ cancels the one appearing in the $Z_{N=8}$ and $Z^A_{N=4}$ partition function. The induced B-constraint for the massless states is

$$
(-)^{h_1+\overline{a}} = +1
$$

eliminating all massless states of $\mathcal{M}^{N=4}_8$ with $(-)^{h_1+\overline{a}} = -1$. The remaining states are then:

Massless spectrum of the magic $\mathcal{M}_3^{N=2}$.

$\hat{\emptyset}$ *super-sector,* $(h_1 = 0,\ \overline{a} = 0)$

$$\left[(\psi^\mu, \chi^{1,2}) \,\oplus\, sp(\psi^\mu, \chi^{1,2})_- sp(\chi^{3,4,5,6})_+\right] \otimes \left[\overline{\psi}^\mu\right], \tag{5.21}$$

1 graviton, 2 gravitini, 2 spin-1/2 fermions, 2 gauge bosons and 2 scalars.

$\hat{\overline{b}}'_1\hat{\overline{b}}'_2$ *super-sector,* $(h_1 = 0,\ \overline{a} = 0)$

$$\begin{aligned}\left[(\psi^\mu, \chi^{1,2}) \,\oplus\, sp(\psi^\mu, \chi^{1,2})_- sp(\chi^{3,4,5,6})_+\right]& \\ \otimes \left[sp(\overline{\chi}^{1,2})_+ sp(\overline{\chi}^{3,4})_+ sp(\overline{y}^{1,2})_+ sp(\overline{y}^{3,4})_+\right]&,\end{aligned} \tag{5.22}$$

4 spin-1/2 fermions, 2 gauge bosons and 4 scalars.

$\hat{b}'_1\hat{\overline{b}}'_1$ *super-sector,* $(h_1 = 1,\ \overline{a} = 1)$

$$\begin{aligned}\left[(sp(\psi^\mu, \chi^{1,2})_- sp(y^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(y^{3,4,5,6})_+\right]& \\ \otimes \left[sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{1,2})_+ sp(\overline{y}^{3,4})_+ sp(\overline{y}^{5,6})_+\right]&,\end{aligned} \tag{5.23}$$

4 spin-1/2 fermions, 2 gauge bosons and 4 scalars.

$\hat{b}'_1\hat{\overline{b}}'_2$ *super-sector,* $(h_1 = 1,\ \overline{a} = 1)$

$$\begin{aligned}\left[(sp(\psi^\mu, \chi^{1,2})_- sp(y^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(y^{3,4,5,6})_+\right]& \\ \otimes \left[sp(\overline{\psi}^\mu)_- sp(\overline{\chi}^{3,4})_+ sp(\overline{y}^{1,2})_+ sp(\overline{y}^{5,6})_+\right]&,\end{aligned} \tag{5.24}$$

4 spin-1/2 fermions, 2 gauge bosons and 4 scalars.

$\hat{A}\hat{b}'_1\hat{\overline{b}}'_1\hat{\overline{b}}'_2$ *super-sector* $(h_1 = 1,\ \overline{a} = 1)$

$$\begin{aligned}\left[(sp(\psi^\mu, \chi^{1,2})_- sp(w^{3,4,5,6})_+ \,\oplus\, sp(\chi^{3,4,5,6})_+ sp(w^{3,4,5,6})_+\right]& \\ \otimes \left[sp(\overline{\psi}^\mu)_s p(\overline{\chi}^{5,6})_+ sp(\overline{y}^{1,2})_+ sp(\overline{y}^{3,4})_+\right]&.\end{aligned} \tag{5.25}$$

4 spin-1/2 fermions, 2 gauge bosons and 4 scalars.

The $N = 2$ graviton multiplet comes from the $\hat{\emptyset}$ super-sector. The same super-sector contains one vector multiplet as well. There are eight additional vector multiplets from the other super-sectors. In total the number of scalars is 18 and they parameterize the Magic $N = 2$ manifold

$$\mathcal{M}_3 = \frac{\mathrm{SU}(3,3)}{S(\mathrm{U}(3) \times \mathrm{U}(3))}, \tag{5.26}$$

This manifold is based to the $N = 2$ holomorphic prepotential

$$F(Z_0, Z^{ij}) = -i\,\frac{\mathrm{Det}(Z^{ij})}{Z_0} = -iZ_0^2\,\mathrm{Det}(t^{ij})\,, \tag{5.27}$$

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where the 3×3 matrix Z^{ij} parameterizes the nine complex scalars ($t^{ij} = Z^{ij}/Z_0$). The Kähler potential associated to the magic $\mathcal{M}_3^{N=2}$ is:

$$K = -\log \; i\mathrm{Det}(t^{ij} - \bar{t}^{ij}) \tag{5.28}$$

and has the property to be identical to the $N = 3$ supergravity coupled to three extra vector multiplets.

Utilizing the same basis sets as in the 4d construction of $\mathcal{M}_3^{N=2}$ and performing similar operations it is straightforward to define the reduced theory in three space time dimensions. Via 3d duality transformation acting on 3d vectors the 3d manifold is extended to: **18** (4d-scalars) + **18** (4d-vectors) + **2** (4d-graviphoton) + **2** (3d-graviphoton, $g_{3,\mu}$) = **40**. The obtained 3d theory is that of the magic square with scalar manifold

$$\mathcal{M}_3^{D=3} = \frac{E_{6(2)}}{\mathrm{SU}(2) \times \mathrm{SU}(6)}\,, \tag{5.29}$$

as expected by the $c-$map operation.

One expects via *oxidation* to five space-time dimensions to construct the scalar manifold of the magic square with 8 5d-scalars:

$$\mathcal{M}_3^{D=5} = \frac{\mathrm{SL}(3, C)}{\mathrm{SU}(3)}\;. \tag{5.30}$$

Although this operation looks straightforward in field theory set-up there is an obstruction in the above stringy construction where all internal left-moving coordinates are twisted. It is therefore impossible to construct this model in our set-up. This obstruction however does not prevent the stringy existence of $\mathcal{M}_3^{D=5}$ via asymmetric orientifold construction or else which appear non-perturbative from the type II "close strings" framework. Hopefully, we will return and try to clarify this obstruction in near future.

6. Discussion

Several type II superstring vacua with $N = 2$ supersymmetry can be constructed. In this work we focussed our attention on those which do not contain in their massless spectrum any hypermultiplet. In this class of vacua the scalar manifold of the vector supermultiplets is always Kähler. It is interesting that for all four dimensional hyper-free constructions, the internal compactification is necessarily *not a Calabi-Yau manifold* or more generally the world-sheet superconformal symmetry is not based on $\mathcal{N} = (2, 2)$ but rather on $\mathcal{N} = (4, 1)$. Indeed, their constructions is left-right asymmetric and can be realized by asymmetric orbifolds via 2d-fermionic construction.

The minimal hyper-free theory with only one massless vector multiplet has been constructed. This theory contains a single *minimally coupled vector multiplet S* associated to the axion-dilaton pair. This theory is exotic from the viewpoint of Calabi-Yau compactification where the vector multiplets are conformaly (non-minimally) coupled. Furthermore, this theory is universal in the sense that is a part of the spectrum of all the other more complex models.

Among the hyper-free $N = 2$ theories two of them are special. They are known to be in correspondence with the symmetric spaces of the "magic square" and furthermore are associated to the Jordan algebras $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{H}}$. They have the additional remarkable property to share the same scalar field content as some supergravity models with more supersymmetry, $N = 6$ and $N = 3$. The one associated to the $N = 6$ turns out to be very special since not only the scalar degrees of freedom but all the bosonic massless degrees of freedom are the same as the $N = 6$ supergravity theory.

The superstring realization of the two $N = 2$ of the magic square turns out to be non-trivial. We were able to construct them by introducing a "twisting mechanism" that eliminates the extra gravitini of the $N = 3$ and $N = 6$ supergravities and creates at the same time the extra spin-$\frac{1}{2}$ fermions and spin-1 gauge bosons that are necessary to balance the $N = 2$ boson-fermion degeneracy.

The "twisting mechanism" is interesting by itself. It is a well defined operation in string theory and is based on "holomorphic Z_2-orbifold stringy constructions":

(i) The four left-moving gravitini are reduced to two as usually by a Z_2-projection.

(ii) Due to the *holomorphic structure* of the projection *two extra gravitini* appears in the "twisted" sector and thus obtain a "twisted $N = (2+2)+4$ realization" of the $N = 8$ supergravity.

(iii) The breaking of the four right-moving supersymmetry is realized via a coupling of the lattice charges to the right-helicity charge $\overline{a}$ and left-twisted charge h, imposing the constrain $(h + \overline{a}) = 0$ *mod* 2. This constrain breaks the two left- and the four right-moving supersymmetry but keeps the gauge bosons and scalars coming from the $N = 2 + 4$ Ramond-Ramond states.

The above three steps define the "twisting mechanism" applied in the case of the magic $N = 2$ associated to the $N = 6$ supergravity. Although this mechanism is well defined in string theory it is not yet known how it could be realized in a Z_2-truncated supergravity for two main reasons:

(i) From where could the twisted gravitini appear?

(ii) How one can define the R-symmetry charges associated to the "twisted states"?

The "twisting mechanism" is even more involved in the case of the $N = 2$ magic associated to the $N = 3$ supergravity coupled to three extra vector multiplets. Here one starts from a "twisted $N = 2 + (1 + 1)$ realization" of $N = 4$ supergravity coupled to eight extra vector multiplets. Then, the constraint $(h + \overline{a}) = 0$ *mod* 2 is introduced which reduces the supersymmetry to $N = 2 + 0$. Here also the construction is "stringy". It is an open interesting problem if an analogous construction can be realized in a Z_2^n-truncated supergravity theories.

By reduction to three space-time dimensions, we are able to construct at the string level two other theories of the magic square namely: $\mathcal{M}_6^{D=3} = \frac{E_{7(-5)}}{\mathrm{SU}(2)\times\mathrm{SO}(12)}$, and $\mathcal{M}_3^{D=3} = \frac{E_{6(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(6)}$.

By oxidation in five space-time dimensions the construction of $\mathcal{M}_6^{D=5} = \frac{SU^*(6)}{USP(6)}$ of the magic square is also achieved at the string level. However, in our stringy set-up, there is an obstruction to define $\mathcal{M}_3^{D=5} = \frac{\mathrm{SL}(3,C)}{\mathrm{SU}(3)}$ since all six right-moving coordinates are twisted and this prevents the oxidation procedure in five-dimensions. On the other hand this obstruction does not implies in general the non-existence at the string level of $\mathcal{M}_3^{D=5}$. It is possible that this theory exists at the string level via other constructions like "asymmetric orientifolds" that may appear as non-petrurbative constructions from "closed string" set-up we had explored here. It remains an open and very interesting question the stringy existence of the other theories of the magic square, in three, four and five space-time dimensions:

$$\begin{array}{lllll}
D=5 & \dfrac{\mathrm{SL}(3,R)}{\mathrm{SO}(3)}\ [5], & \dfrac{\mathrm{SL}(3,C)}{\mathrm{SU}(3)}\ [8], & \dfrac{\mathrm{SU}^*(6)}{\mathrm{USP}(6)}\ [14], & \dfrac{E_{6(-26)}}{F_4}\ [26] \\
D=4 & \dfrac{\mathrm{SP}(6,R)}{\mathrm{U}(3)}\ [12], & \dfrac{\mathrm{SU}(3,3)}{\mathrm{U}(1)\times\mathrm{SU}(3)\times\mathrm{SU}(3)}\ [18], & \dfrac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}\ [30], & \dfrac{E_{7(-25)}}{\mathrm{U}(1)\times E_6}\ [54] \\
D=3 & \dfrac{F_{4,4}}{\mathrm{USP}(6)\times\mathrm{SU}(2)}\ [28], & \dfrac{E_{6(2)}}{\mathrm{SU}(2)\times\mathrm{SU}(6)}\ [40], & \dfrac{E_{7(-5)}}{\mathrm{SO}(12)\times\mathrm{SU}(2)}\ [64], & \dfrac{E_{8(-24)}}{\mathrm{SU}(2)\times E_7}\ [112]
\end{array}$$

Indeed, the explicit constructions at the string level of the magic $N=2$ theories extends the validity at the string level of the entropy formulas obtained via the BPS and non-BPS attractor mechanism introduced in refs [19].

Finally, it will be very interesting to study supersymmetric string vacua and their effective supergravity theories that will be eventually obtained via a "generalized twisting mechanism"; not only in type II theories, but also in heterotic as well as in type II orientifolds with brane and fluxes.

After the submission of this work an interesting paper appeared, by M. Bianchi and S. Ferrara [20], where two other magic square $N_4 = 2$ theories are obtained via an asymmetric orientifold construction, namely

$$\mathcal{M}_7 = \frac{\mathrm{E}_{7(-25)}}{\mathrm{U}(1)\times\mathrm{E}_6}$$

in four space-time dimensions and

$$\mathcal{M}_8 = \frac{\mathrm{E}_{8(-24)}}{\mathrm{SU}(2)\times\mathrm{E}_7}$$

in three dimensions. Even more interesting is the simultaneous appearance in their construction of the two magic scalar manifolds in the same four dimensional $N=2$ theory, with $\mathcal{M}_7$ as the scalar manifold of the vector multiplets and $\mathcal{M}_8$ the one of the hypermultiplets. In three space time dimensions one obtains a *double magic theory* based on $\mathcal{M}_8 \times \mathcal{M}_8$, since $\mathcal{M}_8$ is derived by $\mathcal{M}_7$ via a three dimensional c-map. This doubling of the manifold is absent in our hyper-free construction. In the same work M. Bianchi and S. Ferrara underlined the importance of string magic theories for the validity of the entropy formulae obtained via the BPS and non-BPS attractor mechanism, (for an updated view of the subject, see [21]).

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